→ Conservation Laws in MHD

– here discuss: conservation $\begin{cases} \text{momentum} \\ \text{energy} \\ \text{angular momentum} \end{cases}$

and virial theorems

→ Momentum → key: construct evolution of momentum density

have: $\rho\left(\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V}\right) = -\nabla\left(P + \frac{B^2}{8\pi}\right) + \frac{\underline{B}\cdot\nabla\underline{B}}{4\pi} + \underbrace{\rho\underline{g}}_{\text{body force}}$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underline{V}) = 0$$

$$\Rightarrow \underbrace{\frac{\partial}{\partial t}(\rho\underline{V})}_{\text{momentum density}} + \nabla\cdot\Big(\underbrace{\rho\underline{V}\,\underline{V}}_{\text{Reynolds stress tensor}}\Big) = -\nabla\left(P + \frac{B^2}{8\pi}\right) + \underbrace{\nabla\cdot\frac{\underline{B}\,\underline{B}}{4\pi}}_{\text{Maxwell stress tensor}} + \rho\underline{g}$$

$\underline{\underline{T}}_R = \rho\underline{V}\,\underline{V}$

$T_B = \frac{B^2}{8\pi}\underline{\underline{I}} - \frac{\underline{B}\,\underline{B}}{4\pi}$

thus re-write:

$$\frac{\partial}{\partial t}(\rho\underline{V}) = -\nabla\cdot\underline{\underline{T}} + \rho\underline{g}$$

where

$$\underline{T} = \left(p + \frac{B^2}{8\pi}\right)\underline{\underline{I}} + \frac{\underline{B}\,\underline{B}}{4\pi} - \rho\,\underline{v}\,\underline{v}$$

$$T_{ij} = \left(p + \frac{B^2}{8\pi}\right)\delta_{ij} + \frac{B_i B_j}{4\pi} - \rho v_i v_j$$

↦ ala Gaussian surface

Then, if consider a 'blob' of $\left\{\begin{array}{l}\text{plasma}\\ \text{magnetofluid}\end{array}\right.$:

surface S — crucial

momentum density

$$\frac{\partial}{\partial t}\underbrace{\underline{P}}_{\text{momentum}} = \int d^3x\, \frac{\partial}{\partial t}(\rho\underline{v})$$

blob enclosed by arbitrary, non-dynamical surface.

$$= -\int d^3x\, \nabla\cdot\underline{\underline{T}} + \underbrace{\int d^3x\, \rho\underline{g}}_{\text{net body force}}$$

$$= -\int d\underline{S}\cdot\underline{\underline{T}} + \int d^3x\, \rho\underline{g}$$

So, apart from volume integrated body force,

$$\frac{\partial P}{\partial t} = -\int d\underline{S}\cdot\underline{\underline{T}}$$

$\left\{\begin{array}{l}\text{change in momentum set}\\ \text{by stress on}\\ \text{surface of blob}\end{array}\right.$

$$\underline{\underline{T}} = \left(p + \frac{B^2}{8\pi}\right)\underline{\underline{I}} - \frac{\underline{B}\,\underline{B}}{4\pi} + \rho\,\underline{v}\,\underline{v}$$

Thus, can identify ways momentum is lost by the blob:

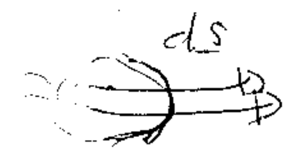
$-\underline{\underline{T}}_R \cdot d\underline{S} = -\rho \underline{V}\,\underline{V} \cdot d\underline{S}$ → flux of momentum density thru surface

$-\underline{\underline{T}}_{P_{tot}} \cdot d\underline{S} = -\left(P + \frac{B^2}{8\pi}\right) \cdot d\underline{S}$ → pressure (total) force on surface, in $-d\underline{S}$ direction

$-\underline{\underline{T}}_{\substack{Mag\\ten}} \cdot d\underline{S} = \frac{\underline{B}}{4\pi}\, \underline{B} \cdot d\underline{S}$ → magnetic tension force in $+\underline{B}$ direction, piercing surface

$\sim (\underline{B} \cdot d\underline{S})\ \frac{\underline{B}}{4\pi}$ → tension of $\underline{B}/4\pi$, per line

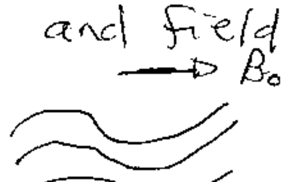
$(\underline{B} \cdot d\underline{S})$ → # of lines thru $d\underline{S}$ outward

i.e. 

→ Note that magnetic tension is independent of sign of $\underline{B}$ (as it should, tension is, strictly speaking, a dyad $\updownarrow$, not $\uparrow$)

→ tensor field $\sim \underline{B}\,\underline{B}$

→ can make obvious analogy between 'strings' and field lines



Alfvén wave

strings/area $= B$

$\sigma = \rho/B$ → mass per length, of string

$T = B/4\pi$

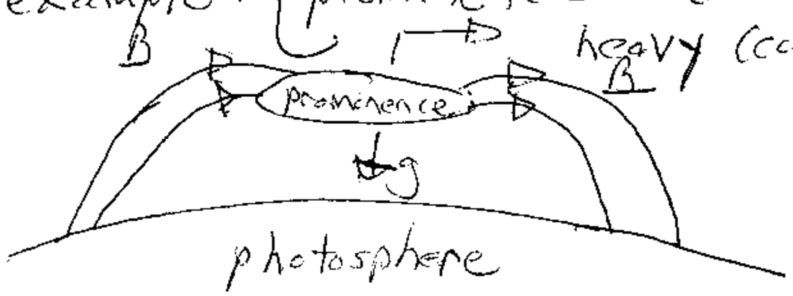
$V_{ph}^2 = T/\sigma$

$= B^2/4\pi\rho_0$

$= V_A^2$

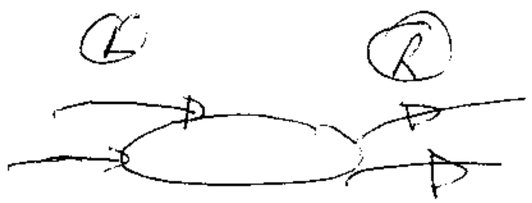
→ example: { Solar prominence } (see cover of Kulsrud)

→ heavy (cool) → requires support against gravity



{ prominences often associated with radiative condensation

i.e.



$L \Rightarrow$ #lines/area $= \underline{B}\cdot d\underline{S} < 0$ (inward)

force/Line is toward $\downarrow$

$\therefore$ $\underline{F}_{L} \rightarrow$ toward upper left

$R \Rightarrow$ # lines/area $= \underline{B}\cdot d\underline{S} > 0$

f/Line is toward upper right

$F_{R} \rightarrow$ toward upper right

thus → prominence supported by magnetic tension (ala hammock — string)

→ squashing $\underline{B}$ → support by magnetic pressure, too . . . !

→ The Skeptic: "What of EM Momentum?"

$$\underline{P}_{EM} = \underline{E}\times\underline{B}/4\pi c$$

$$E \sim \frac{VB}{c} \Rightarrow P_{EM} \sim (\rho V)\, B^2/4\pi\rho c^2$$

$$\sim \rho V \left(V_A^2/c^2\right) \ll 1$$

n.b. obviously important in relativistic and EMHD! for $V_A \ll c$.

→ Angular Momentum → need Kulsrud ---- "virtual surface"

→ Energy

kinetic thermal magnetic gravity

Now energy: $E = E_V + E_p + E_B + E_g$

$$E = \int_V d^3x \left[\frac{1}{2}\rho V^2 + \frac{p}{\gamma-1} + \frac{B^2}{8\pi} + \frac{\rho\phi}{2}\right]$$

self-gravitating blob

where $\begin{cases} \underline{g} = -\nabla\phi \\ \nabla^2\phi = 4\pi G\rho \end{cases}$ i.e. $\underline{g}$ evolves self-consistently (not "constant")

N.B. Problem: Jeans Instability

→ Calculate the growth rate of density perturbations in an un-magnetized, self-gravitating fluid

→ repeat in 1D using Vlasov equation

→ Where does E_p come from?

Consider work to compress plasma/fluid, i.e.

$$dW = -p\,dV$$

$$\Delta E = -\int_0^{\rho_0} P(\rho)\, d(1/\rho) = \int_0^{\rho_0} (\rho/\rho_0)^{\gamma} P_0 \frac{d\rho}{\rho^2}$$

$$= \frac{P_0}{\rho_0(\gamma-1)} \Rightarrow \underset{\substack{\downarrow \\ \text{energy density}}}{\mathcal{E}} = \rho_0 \Delta E = \frac{P_0}{(\gamma-1)}$$

→ for energy balance, crank it out, using MHD equations ....

$$\frac{dE}{dt} = \underset{①}{\frac{d E_V}{dt}} + \underset{②}{\frac{d}{dt} E_p} + \underset{③}{\frac{d}{dt} E_B} + \underset{④}{\frac{d}{dt} E_g}$$

① $$\frac{d}{dt} E_V = \int d^3x \frac{\partial}{\partial t}\left(\frac{\rho \underline{V}^2}{2}\right)$$

$$= \int d^3x \left[V^2 \frac{\partial \rho}{\partial t} + \rho \frac{\partial \underline{V}}{\partial t} \cdot \underline{V} \right] d^3x$$

↳ i b p leaves S.T. and cancels 2nd.

$$= \int d^3x \left[-\frac{\underline{V}^2}{2} \nabla\cdot(\rho \underline{V}) - \underline{V}\cdot \rho(\underline{V}\cdot\nabla\underline{V}) - \underline{V}\cdot\nabla p + \underline{V}\cdot(\underline{J}\times\underline{B}) - \rho \underline{V}\cdot\nabla\phi \right]$$

i.e. $\int -\frac{\underline{V}^2}{2}\,\nabla\cdot(\rho\underline{V}) = -\frac{V^2}{2}\rho\underline{V}\Big| + \int(\underline{V}\cdot\nabla\underline{V})\cdot\rho\,\underline{V}$

cancels 2nd term in $\frac{dE_V}{dt}$

② $$\frac{d}{dt}E_p = \int\frac{d^3x}{\gamma-1}\,\frac{\partial p}{\partial t}$$

Now eqn. state $\Rightarrow$ $$\frac{1}{p}\frac{dp}{dt} + \frac{\gamma}{\rho}\frac{d\rho}{dt} = 0$$

$$\left[\frac{1}{(p/\rho^\gamma)}\frac{d}{dt}(p/\rho^\gamma) = 0\right]$$

and $$\frac{1}{\rho}\frac{d\rho}{dt} = -\nabla\cdot\underline{V}$$

$$\Rightarrow \frac{\partial p}{\partial t} = -\underline{V}\cdot\nabla p - \gamma\, p\,\nabla\cdot\underline{V}$$

So $$\frac{d}{dt}E_p = \frac{-1}{(\gamma-1)}\int d^3x\,(\underline{V}\cdot\nabla p + \gamma p\,\nabla\cdot\underline{V})$$

$$= -\int d^3x\left[\underbrace{\frac{\gamma}{\gamma-1}\nabla\cdot(p\underline{V})}_{\text{yields a surface term}} - \underbrace{\underline{V}\cdot\nabla p}_{\text{cancels } \underline{V}\cdot\nabla p \text{ term in } \frac{dE_n}{dt}}\right]$$

expect similar relation between $\underline{J}\times\underline{B}$ and $\frac{\partial}{\partial t}B^2$ .... so...

③ $$\frac{d}{dt}E_B = \frac{1}{4\pi}\int d^3x\ \underline{B}\cdot\frac{\partial\underline{B}}{\partial t}$$

$$= \frac{1}{4\pi}\int d^3x\ \underline{B}\cdot\left(\underline{\nabla}\times\underline{v}\times\underline{B}\right)$$ by induction eqn.

(a) (b)

$$= -\int d^3x\left\{\underbrace{\underline{\nabla}\cdot\left[\frac{\underline{B}\times(\underline{v}\times\underline{B})}{4\pi}\right]}_{\substack{\text{surface term}\\(\to\text{Poynting})}} - \underbrace{\frac{(\underline{\nabla}\times\underline{B})}{4\pi}\cdot(\underline{v}\times\underline{B})}_{\underline{J}\cdot\underline{v}\times\underline{B}}\right\}$$

$$\Rightarrow = \int d^3x\ \underline{J}\cdot(\underline{v}\times\underline{B}) = -\int d^3x\ \underbrace{(\underline{J}\times\underline{B})\cdot\underline{v}}_{\substack{\text{cancels } \underline{v}\cdot\underline{J}\times\underline{B} \text{ term}\\ \text{in } dE_V/dt}}$$

which leaves:

④ $$\frac{dE_g}{dt} = \frac{1}{2}\int d^3x\left(\phi\frac{\partial\rho}{\partial t} + \rho\frac{\partial\phi}{\partial t}\right)$$

$$= \frac{1}{2}\int d^3x\ \phi\frac{\partial\rho}{\partial t} + \int d^3x\ \frac{\nabla^2\phi}{8\pi G}\frac{\partial\phi}{\partial t}$$ ibp ⇒

$$= \frac{1}{2}\int d^3x\,\phi\frac{\partial\rho}{\partial t}d^3x + \int\frac{\phi}{8\pi G}\nabla^2\frac{\partial\phi}{\partial t}d^3x$$

or $$\frac{dE_g}{dt} = \frac{1}{2}\int \phi \frac{\partial \rho}{\partial t} d^3x + \frac{1}{2}\int d^3x\, \phi \frac{\partial \rho}{\partial t}$$

$$= \int d^3x\, \phi \frac{\partial \rho}{\partial t} = -\int d^3x\, \phi\, \nabla\cdot(\rho \underline{V})$$

$$= + \int d^3x\, \rho \underline{V}\cdot\underline{\nabla}\phi$$

cancels $-\rho \underline{V}\cdot\nabla\phi$ in $\frac{dE_k}{dt}$ $\quad + -\int d\underline{S}\cdot \rho\phi\, \underline{V}$

Note: $-\underline{V}\cdot\underline{\nabla}P$; $\underline{V}\cdot(\underline{J}\times\underline{B})$; $-\rho\underline{V}\cdot\underline{\nabla}\phi$; $\underline{V}\cdot\rho\underline{V}\cdot\underline{\nabla}\underline{V}$

terms all cancel in dE_V/dt !

Now adding up all 4 pieces $\Rightarrow$

$$\boxed{\frac{d}{dt}E = -\int d\underline{S}\cdot\left[\rho\underline{V}\frac{V^2}{2} + \frac{\gamma}{\gamma-1}P\underline{V} - \frac{(\underline{V}\times\underline{B})\times\underline{B}}{4\pi} + \rho\underline{V}\phi\right]}$$

i.e. not surprisingly, only survivors are surface terms..... $\Rightarrow$ in ideal MHD, only change in energy of blob involves boundary...

i.e. have:

$$\frac{dE}{dt} = -\int d\underline{S} \cdot \left[\overset{①}{\rho \underline{V} \frac{V^2}{2}} + \overset{②}{\frac{\gamma}{\gamma-1} p \underline{V}} - \overset{③}{\frac{(\underline{V}\times\underline{B})\times\underline{B}}{4\pi}} + \overset{④}{\rho \underline{V}\phi} \right]$$

① → kinetic energy loss via simple kinetic energy flow thru surface.

② → $-\frac{\gamma \underline{V} \cdot d\underline{S}\, p}{\gamma - 1}$ ⟷ outward flow of enthalpy

i.e. $-\frac{\gamma\, p \underline{V} \cdot dS}{\gamma-1} = \underbrace{-\frac{p}{\gamma-1}\underline{V}\cdot d\underline{S}}_{\text{outward flow of thermal energy } \left(d\underline{S}\cdot \underline{V}\frac{p}{\gamma-1}\right) \text{ thru } S} \underbrace{-p\,\underline{V}\cdot dS}_{p\,dV \text{ work of blob on exterior}}$

why the γ?! →

③

as $\underline{E} = -\frac{\underline{V}\times\underline{B}}{c}$ . . . →

so ③ $= d\underline{S} \cdot \frac{\underline{E}\times\underline{B}}{4\pi c}$ → $\left\{ \text{loss of energy by Poynting flux} \right.$

④ loss of gravitational potential energy due outflow from blob ...

It's all clear!!

this brings us to ....

→ <u>Virial Theorems in MHD</u>

- what is a virial theorem

- why yet another theorem?

→ Virial Theorems are:

- space/time averaged energy theorems

- "lumped parameter" relations for energies in complex, multi-element interacting systems

- useful for 'back-of-envelope' estimates, etc.

- logically extend the moment program:

$$f(\underline{x}, \underline{v}, t) \xrightarrow[\underline{v}\text{ moments}]{} n(\underline{x}, t), \underline{v}, T \xrightarrow[\text{virial integrals}]{} E_v, E_B, \text{etc.}$$

phase space fluid — position space fluid — #

space integrals

Before proceeding:

¿ Can an isolated blob of MHD plasma confine itself without self gravity ?

Easily answered by Virial Theorem....

Recall, for system of particles, Virial theorem derived by considering:

$$\frac{d}{dt}\left(\sum_i \underline{p}_i \cdot \underline{x}_i\right) = \sum_i \underline{p}_i \cdot \dot{\underline{x}}_i + \sum_i \dot{\underline{p}}_i \cdot \underline{x}_i$$

$$= \underbrace{2T}_{\text{kinetic energy}} + \underbrace{\sum_i \left(-\frac{\partial U}{\partial \underline{x}_i}\right) \cdot \underline{x}_i}_{\text{via Newton's Law}}$$

Now, if $\sum_i \underline{p}_i \cdot \underline{x}_i$ bounded,

$$\left\langle \frac{d}{dt} \sum \underline{p}_i \cdot \underline{x}_i \right\rangle = \frac{1}{T} \int_0^T dt \frac{d}{dt}\left(\sum_i \underline{p}_i \cdot \underline{x}_i\right)$$

$$\xrightarrow[T \to \infty]{} 0$$

so ...

→ (first) Virial of system

$$2\langle T\rangle = \left\langle \sum_i \frac{\partial U}{\partial \underline{x}_i}\cdot \underline{x}_i \right\rangle$$

Further if $U = U(\underline{x}_1, \underline{x}_2, \ldots \underline{x}_N)$

where $U(\alpha\underline{x}_1, \alpha\underline{x}_2, \ldots \alpha\underline{x}_N) = \alpha^k U(\underline{x}_1, \underline{x}_2, \ldots \underline{x}_N)$

(scaling ⟺ structure of power-law potentials → i.e. h.o. → $k=2$, Coulomb → $k=-1$)

→ homogeneous function

$$\Rightarrow \boxed{2\langle T\rangle = k\langle U\rangle}$$

but of course:

$$T + U = \langle T\rangle + \langle U\rangle = E$$

then $\left(\frac{k}{2}+1\right)\langle U\rangle = E$

$$\boxed{\langle U\rangle = \frac{2}{k+2}E}, \quad \langle T\rangle = \frac{kE}{k+2}$$

check: $k=2$, $\langle U\rangle = \frac{1}{2}E$, $\langle T\rangle = \frac{1}{2}E$ ✓

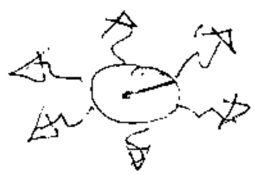
$k=-1$, $\langle T\rangle = -E$ ✓ $\Rightarrow$ (bounded motion only if total energy negative (i.e. bound state))

$(\Rightarrow E<0)$

Aside: Simplest realization of negative specific heat 'paradox', i.e.

 $\rightarrow$ consider 'blob' of self gravitating matter

$$E \sim -1/R$$

if radiation  $\rightarrow$ E decreases $\leftrightarrow$ R decreases

$\therefore$ $(-E)$ increases $\Rightarrow$ $\langle T \rangle$ increases
$\hookrightarrow$ kinetic energy

but $\langle T \rangle \sim$ temperature, so have cycle of: radiative cooling $\Rightarrow$ temperature increase !?

$$\Rightarrow \underset{\text{specific heat}}{c} < 0 \;!?$$

In the days before the discovery of nuclear fusion, this was thought to be what 'heated' stars. Kelvin, in particular, was a proponent.

Now, proceeding to full Virial theorem ...

→ Consider equation of motion

$$\underbrace{\frac{\partial}{\partial t}(\rho v_i)}_{\text{momentum}} = -\frac{\partial}{\partial x_j} \underbrace{T_{ij}}_{\underline{\text{full}}\text{ stress tensor}}$$

$$T_{ij} = \rho v_i v_j + \left(p + \frac{B^2}{8\pi}\right)\delta_{ij} - \frac{B_i B_j}{4\pi} + \rho \phi \delta_{ij}$$

Now, recalling relation of Virial to $\frac{d}{dt}(\underline{p}\cdot\underline{x})$
⇒ consider:

$$I_{ij} = \int d^3x \, \rho x_i x_j \qquad (\sim \text{moment of inertia})$$

↦ Virial theorem is for tensor...

and

$$\frac{d}{dt} I_{ij} = \int d^3x \, \frac{\partial \rho}{\partial t} x_i x_j$$

$$= -\int d^3x \, \frac{\partial}{\partial x_t}(\rho v_t)\, x_i x_j$$

integrating by parts assuming ρ compact (i.e. 'blob' of interest)

$$= \int d^3x \left[\rho x_i v_j + \rho x_j v_i\right]$$

so

$$\frac{d^2 I_{ij}}{dt^2} = \int d^3x \left[x_i \left(\frac{\partial}{\partial t}\rho v_j\right) + x_j \frac{\partial}{\partial t}(\rho v_i) \right]$$

but $$\frac{\partial}{\partial t}(\rho V_i) = -\frac{\partial}{\partial x_k} T_{ik}$$

$\Rightarrow$

$$\frac{d^2 I_{ij}}{dt^2} = -\int d^3x \left[x_i \frac{\partial T_{j,t}}{\partial x_t} + x_j \frac{\partial T_{i,t}}{\partial x_t} \right]$$

and integrating by parts, assuming $\left\{ \begin{array}{l} \text{compact blob,} \\ \text{no external} \\ \text{linkage} \end{array} \right.$

$\Rightarrow$

$$\frac{d^2 I_{ij}}{dt^2} = +\int d^3x \left[\delta_{i,t} T_{j,t} + \delta_{j,t} T_{i,t} \right]$$

$\partial x_i / \partial x_t = 0$
unless $i = t$

$$= +\int d^3x \left[T_{j,i} + T_{i,j} \right]$$

and as T_{ij} manifestly symmetric $\Rightarrow$

$$\boxed{\begin{aligned} &\frac{1}{2} \frac{d^2 I_{ij}}{dt^2} = +\int d^3x \, T_{ij} \\ &T_{ij} = \rho V_i V_j + \left(p + \frac{B^2}{8\pi} \right) \delta_{ij} - \frac{B_i B_j}{4\pi} + \rho \phi \delta_{ij} \end{aligned}}$$

— tensor virial theorem.

note unlike simple pt particle example, time dependence remains.

Now, to make contact with notions of energy, etc., useful to contract the tensor

$$I = I_{ii} = \mathrm{tr}\, I_{ij}$$

repeated indexes summed

$\mathrm{tr}\,(\text{V.T.}) \Rightarrow$

$$\mathrm{tr}\, \frac{1}{2}\frac{d^2 I_{ij}}{dt^2} = \frac{d^2}{dt^2}\left(\int d^3x\, \frac{\rho x^2}{2}\right)$$

$$= \mathrm{tr} \int d^3x \left[\rho v_i v_j + \left(p + \frac{B^2}{8\pi}\right)\delta_{ij} - \frac{B_i B_j}{4\pi} + \rho\phi\, \delta_{ij}\right]$$

$$= \int d^3x \left[\rho v^2 + 3\left(p + \frac{B^2}{8\pi}\right) - \frac{B^2}{4\pi} + 3\rho\phi\right]$$

$\therefore \quad I \equiv \int d^3x\, \rho x^2/2 \quad \Rightarrow$

$$\boxed{\frac{d^2 I}{dt^2} = \int d^3x \left[\rho v^2 + 3p + \frac{B^2}{8\pi} + 3\rho\phi\right]}$$

$\rightarrow$ Scalar Virial Theorem.

Now, first neglect self-gravitation $\Rightarrow$

$$\frac{d^2 I}{dt^2} = \frac{d^2}{dt^2}\left(\int d^3x \, \frac{\rho x^2}{2}\right)$$

$$= \int d^3x \left[\rho V^2 + 3p + B^2/8\pi\right]$$

Now $\rightarrow$ can an isolated blob of MHD fluid confine itself ??

If 'self-confined' $\Rightarrow$ $\frac{dI}{dt} \leq 0$

$\frac{d^2 I}{dt^2} \leq 0$

i.e. quiescent $\Rightarrow \dot{I}, \ddot{I} = 0$

stable pulsation $\Rightarrow \ddot{I} = -\Omega^2 I < 0$

but have $\ddot{I} = \int d^3x \left[\rho V^2 + 3p + B^2/8\pi\right]$

so even if $V^2 = 0$ (no fluid motion in blob) $\Rightarrow$

$p > 0$, $B^2/8\pi > 0$ $\Rightarrow \ddot{I} > 0$!

$\therefore$ No $\rightarrow$ isolated blob can't confine itself.

More generally, noting that

$$E_V = \int d^3x\, \rho V^2/2$$

$$E_P = \int d^3x \frac{P}{\gamma - 1} = \frac{3}{2}\int d^3x\, P \qquad \text{(gas)}$$

$$E_B = \int d^3x \frac{B^2}{8\pi}$$

can write scalar Virial theorem in form:

$$\boxed{\frac{d^2 I}{dt^2} = 2E_V + 2E_P + E_B}$$

simple relation in terms energies.

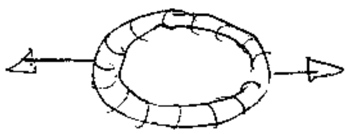
Aside: $\Rightarrow$ So, isolated blob can't confine itself

$\Rightarrow$ how is $\left\{\begin{array}{l} \text{tokamak} \rightarrow B_T \text{ for } \left\{\begin{array}{l}\text{stability; not} \\ \text{transport}\end{array}\right. \\ \underline{\text{or}} \text{ - better} \quad \text{macro-confinement} \\ \text{RFP} \rightarrow \text{weak external } B_T \text{ guide} \\ \qquad \text{(negligible)} \end{array}\right.$

confined ?? Confinement by wall is unacceptable ...

x

Answer: → toroidal plasma tends to expand toroidally



→ held in place by { conducting shell (often undesireable) or "vertical field" }

e.g.

→ additional external P_{Mag} to oppose toroidal expansion – vertical field

→ image currents in close-in conducting shell can do likewise

JET anecdote re: vertical field failure ...

Now, retaining self-gravitation:

$$T_{ij}\Big|_{\text{gravity}} = \rho\phi\,\delta_{ij} = 2\underbrace{\left(\frac{\rho\phi}{2}\right)}_{\mathcal{E}_{\text{gravity}}}\delta_{ij}$$

to calculate:

$$\nabla^2\phi = 4\pi G\rho$$

$$\Rightarrow \quad \phi = -G\int d^3x\,\frac{\rho(x')}{|x-x'|}$$

so

$$T_{ij}\Big|_{\text{gravity}} = T\Big|_{\text{gravity}} \delta_{ij}$$

$\Rightarrow$

$$T = -\frac{G}{2}\int d^3x \int d^3x' \, \frac{\rho(\underline{x})\,\rho(\underline{x}')}{|\underline{x}-\underline{x}'|}$$

$$= +E_{\text{gravitation}} \equiv \quad E_g < 0$$

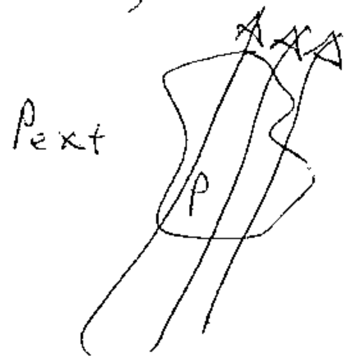
so scalar Virial theorem becomes, with gravity $\Rightarrow$

$$\boxed{\frac{1}{2}\frac{d^2 I}{dt^2} = 2E_v + 2E_p - |E_g| \quad + E_B}$$

so with gravity, can have self-confining blob (no surprise...)

This brings us to another application of Virial theorems, namely proto-stellar cloud collapse....

— now, consider a plasma cloud/blob



- mass M, radius R
- threaded by B
- pressure p, external pressure P_0
- no bulk motion
- frozen flux

now, easy to show for $\ddot{I} = 0$, $\underline{V} = 0$, must have:

surface terms

$$2E_p - |E_g| + E_B = \underbrace{\int dA\, P_{ext}\, \hat{x}\cdot\hat{n}}_{\text{external pressure}} - \underbrace{\int dA\, \underline{x}\cdot\underline{\underline{T}}_B\cdot\hat{n}}_{\text{magnetic stress thru surface (threading fields)}}$$

Now, can estimate:

$M = \int \rho\, dV \rightarrow$ total mass

$E_p \cong c_s^2 M$

$$|E_g| \cong \underbrace{\alpha}_{\text{form factor}} \frac{GM^2}{R}$$

For frozen flux, $\Phi \sim \pi R^2 B$

so $\quad E_B + \int dA\, \underline{x} \cdot \underline{\underline{T}}_B \cdot \hat{n} \sim \beta \Phi^2 / R$

$\Rightarrow$ have: (eliminating extraneous factors):

$$\boxed{R^2 P_{ext} \sim \left(\frac{\beta \Phi^2}{R} - \alpha \frac{G M^2}{R} + \frac{3}{2} c_s^2 M \right)}$$

$\rightarrow$ scalar virial theorem for cloud....

Now: $\quad P_{ext} \sim \left(\frac{\beta \Phi^2}{R^3} - \alpha \frac{G M^2}{R^3} + \frac{3}{2} \frac{c_s^2 M}{R^2} \right)$

$\rightarrow$ if $\Phi, G \rightarrow 0 \quad \rightarrow$ need $P_{int} = P_{ext}$ for confinement.....

$\rightarrow$ if $\Phi = 0$

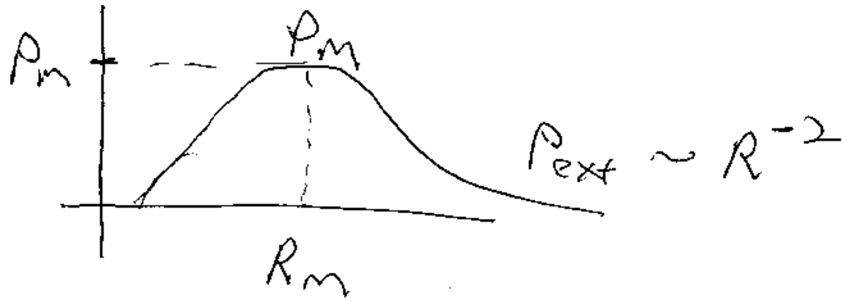
$$P_{ext} = -\frac{\alpha G M^2}{R^3} + \frac{3}{2} \frac{c_s^2 M}{R^2}$$

$dP/dR = 0 \quad \Rightarrow \quad 3 \alpha \frac{G M^2}{R^4} = 3 \frac{c_s^2 M}{R^3}$

$$R_{max} = G M \alpha / c_s^2$$

[Note: $\Rightarrow R_m^2 = (G\rho / c_s^2)^{-1/2} \Rightarrow L_{Jeans}^2$]

so P_{ext} needed



- $P > P_{max}$ $\Rightarrow$ no equilibrium

- $R < R_{max}$ $\Rightarrow$ P_{ext} must decrease to maintain equilibrium $\Rightarrow$ instability to gravitational collapse!

- $\Phi \neq 0$ (magnetic field on ...) $\Rightarrow$ note immediately that magnetic support scales similarly to gravitational attraction

$\Rightarrow$

$$P_{ext} \sim \left[\left(\beta \Phi^2 - \alpha G M^2\right)/R^3 + \frac{3}{2} \frac{c_s^2 M}{R^2} \right]$$

so key point is: $\left(\beta \Phi^2 - \alpha G M^2\right) \lessgtr 0$?!

$$\Rightarrow M \gtrless M_\Phi = \sqrt{\beta/\alpha}\, \Phi / G^{1/2}$$

$M < M_\Phi$ $\Rightarrow$ magnetically subcritical mass for gravitational collapse

$M > M_\Phi \Rightarrow$ magnetically super-critical mass for collapse.

i.e. $M < M_\Phi \Rightarrow$ repulsive effects $\left\{ \begin{matrix} \text{field} \\ \text{thermal} \end{matrix} \right\}$ pressure always win
$(M_\Phi^2 - M^2 > 0)$ $\Rightarrow$ no ammount of external compression can induce indefinite contraction, IF flux remains frozen in

$M > M_\Phi \Rightarrow$ sufficient external pressure/compression can induce gravitational
$(M_\Phi^2 - M^2 < 0)$ collapse, even if flux frozen in.

[Note: IF kinetic energy contribution, NL Alfven waves can support cloud.]

For perspective, recall:

- (famous) Chandrasekhar Mass
  - $M > M_{Chandra} \Rightarrow$ collapse
  - $M < M_{Chandra} \Rightarrow$ no collapse.

$M_{Chandra}$ derived for degenerate Fermi gas equation of state $\rightarrow \gamma = 4/3$, instead of $\gamma = 5/3$.

— of flux-freezing $\Rightarrow$ $\Phi \sim B R^2$
$\rho R^3 \sim M$

$\Rightarrow B \sim R^{-2} \Rightarrow B \sim \rho^{2/3}$

$\therefore B^2 \sim P_{Mag} \sim \rho^{4/3}$

$\Rightarrow$ <u>if</u> flux frozen, field obeys equation of state like Fermi gas

(i.e. flux freezing is akin to exclusion, albeit on field-lines-per-fluid-element)

$\therefore$ an analogue to Chandrasekhar mass seems quite plausible ....

Aside: Chandrasekhar Limit — Simple Derivation
(c.f.: Shapiro, Teukolsky)

→ suppose: N Fermions in star of radius R

$\therefore \quad n_{\text{Fermion}} \sim N/R^3$

$\therefore$ Vol./Fermion $\sim 1/n$ (Pauli exclusion)

$p \sim \hbar/\Delta x \sim \hbar n^{1/3}$ (Heisenberg Uncertainty)

p → Fermion Momentum

⇒ Fermion energy (per Fermion): $E_F = pc \sim \hbar c \dfrac{N^{1/3}}{R}$

replaces: (i.e. Thermal energy)

Gravitational Energy (per Fermion): $E_{Grav} \sim -\dfrac{GM m_b}{R}$

m_b → Baryon mass

$M \sim N \, m_B$

Pressure → electron
Mass → Baryon

$\therefore \quad E = E_F + E_G$

$$= \frac{\hbar c N^{1/3}}{R} - \frac{G N m_B^2}{R}$$

note: $E = E_F + E_G$

$$= \frac{\hbar c N^{1/3}}{R} - \frac{G N m_B^2}{R}$$

$E > 0 \Rightarrow$ decrease E, E_F by increasing R.

but as $E_F \downarrow$, electrons non-relativistic,

$\therefore E_F \sim 1/R^2 \rightarrow$ eqbrm.

$E < 0 \Rightarrow$ decrease E without bound by decreasing $R \Rightarrow$ collapse

$\therefore$ eqbrm: $\hbar c N^{1/3} = G N m_B^2$

$$N_{Max} = \left(\hbar c / G m_B^2\right)^{3/2} \sim 2 \times 10^{57} \quad \text{(proton)}$$

$$\therefore \quad M_{Chandra} = N_{max} m_B \sim 1.5 M_{\odot}$$

→ Magnetic Helicity

– another conserved quantity in ideal MHD is magnetic helicity K

$$K = \int_V d^3x\ \underline{A}\cdot\underline{B}$$

V is taken to be the volume of a 'flux tube'.

– what, yet another invariant!?!

→ K is different ⇒ has topological interpretation

$$K = \int_V d^3x\ A\cdot\underline{\nabla}\times\underline{A}$$

→ $\underline{x} \to -\underline{x}$ flips sign of K

→ K is a pseudo-scalar

∴ has orientation or "handedness"...

Proceed via:

- show K conservation
- discuss interpretation of K
- comment on utility ⇒ Taylor Relaxation

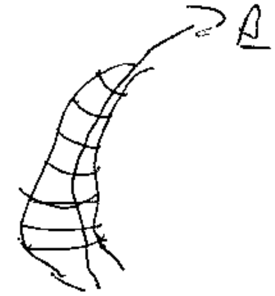
N.B.: Important → K is gauge invariant

i.e. if $\underline{A} \to \underline{A} + \underline{\nabla}\chi$

$$K \to K + \int_V d^3x \; \underline{\nabla}\chi \cdot \underline{B}$$

$$= K + \int_V d^3x \; \underline{\nabla}\cdot(\underline{B}\,\chi)$$

$= 0$, to surface term. $\left\{ \underline{B}\cdot\hat{n} = 0 \right.$ on surface of tube



Now, consider a blob of MHD fluid in motion

can show $\dfrac{dK}{dt} =$

$$\underline{E} + \frac{\underline{V}\times\underline{B}}{c} = \eta\,\underline{J}$$

$$\underline{E} = -\frac{1}{c}\frac{\partial \underline{A}}{\partial t} - \underline{\nabla}\phi$$

$\Rightarrow$

$$\frac{\partial \underline{A}}{\partial t} = \underline{V}\times\underline{\nabla}\times\underline{A} - c\,\underline{\nabla}\phi - c\,\eta\,\underline{J}$$

$$\frac{\partial \underline{B}}{\partial t} = -\underline{V}\cdot\underline{\nabla}\,\underline{B} + \underline{B}\cdot\underline{\nabla}\,\underline{V} - \underline{B}\,\underline{\nabla}\cdot\underline{V} + \eta\,\nabla^2\underline{B}$$

$$\frac{dK}{dt} = \frac{d}{dt}\int_V d^3x \,(\underline{A}\cdot\underline{B})$$

$$= \int d^3x \left(\frac{d\underline{A}}{dt}\cdot\underline{B} + \underline{A}\cdot\frac{d\underline{B}}{dt}\right) + \int \underline{A}\cdot\underline{B}\,\frac{d}{dt}\,d^3x$$

$$\frac{dK}{dt} = \int d^3x \left(\frac{\partial \underline{A}}{\partial t} \cdot \underline{B} + (\underline{V} \cdot \underline{\nabla} \underline{A}) \cdot \underline{B} + \underline{A} \cdot \frac{\partial \underline{B}}{\partial t} + \underline{A} \cdot (\underline{V} \cdot \underline{\nabla}\, \underline{B}) \right) + \underline{A} \cdot \underline{B}\; \underline{\nabla} \cdot \underline{V} \Big)$$

where $\frac{d}{dt} d^3x = \underline{\nabla} \cdot \underline{V}$

i.e. $\frac{d}{dt} dV = \frac{d}{dt} d\underline{\sigma} \cdot d\underline{\ell} + d\underline{\sigma}\, \frac{d}{dt} d\underline{\ell}$

$$= - d\underline{\ell} \cdot \underline{\nabla}\underline{V} \cdot d\underline{\sigma} + (\underline{\nabla} \cdot \underline{V})(d\underline{\sigma} \cdot d\underline{\ell}) + d\underline{\ell} \cdot \underline{\nabla}\underline{V} \cdot d\underline{\sigma}$$

$$= \underline{\nabla} \cdot \underline{V}\; d^3x$$

s.t. and $\underline{B} \cdot \hat{n}$ on surface of tube.

$$\frac{dK}{dt} = \int d^3x \Big[\big(\cancel{\underline{B} \cdot \underline{V} \times \underline{B}} - c\, \cancel{\underline{B} \cdot \underline{\nabla}\phi} - c\eta\, \underline{J} \cdot \underline{B} \big) + \underline{A} \cdot \big(\underline{\nabla} \times (\underline{V} \times \underline{B}) \big) + \underline{\nabla} \cdot \big((\underline{A} \cdot \underline{B}) \underline{V} \big) + \underline{A} \cdot \eta \nabla^2 \underline{B} \Big]$$

where $\underline{A} \cdot (\underline{V} \cdot \underline{\nabla}\, \underline{B}) + \underline{B} \cdot (\underline{V} \cdot \underline{\nabla} \underline{A}) + \underline{A} \cdot \underline{B}\; \nabla \cdot \underline{V} = \nabla \cdot (\underline{V}\, \underline{A} \cdot \underline{B})$

$$\frac{dK}{dt} = \int d^3x \Big[\nabla \cdot \big((\underline{A} \cdot \underline{B}) \underline{V} \big) + \underline{\nabla} \cdot \big((\underline{V} \times \underline{B}) \times \underline{A} \big) + \cancel{(\underline{V} \times \underline{B})} \cdot (\underline{\nabla} \times \underline{A})$$

$$- c\eta\, \underline{J} \cdot \underline{B} - \eta (\underline{A} \cdot \underline{\nabla} \times \underline{J})\, c \Big]$$

$$\Rightarrow \quad \frac{dK}{dt} = \int d^3x \left\{ \underline{\nabla} \cdot \left[(\underline{A} \cdot \underline{B}) \underline{v} + (\underline{v} \times \underline{B}) \times \underline{A} + c\eta (\underline{A} \times \underline{J}) \right] - c\eta\, \underline{J} \cdot \underline{B} - c\eta\, \underline{J} \cdot \underline{B} \right\}$$

$$= \int d\underline{S} \cdot \left[(\underline{A} \cdot \underline{B}) \underline{v} + (\underline{v} \times \underline{B}) \times \underline{A} + c\eta\, \underline{A} \times \underline{J} \right]$$

$$- 2 \int d^3x \left[c\eta\, \underline{J} \cdot \underline{B} \right]$$

$$= \int d\underline{S} \cdot \left[\cancel{(\underline{A} \cdot \underline{B}) \underline{v}}^{\,0} - \cancel{(\underline{A} \cdot \underline{B}) \underline{v}}^{\,0} + \cancel{(\underline{A} \cdot \underline{v}) \underline{B}}_{\,0} \right] - c\eta \int d\underline{S} \cdot \underline{J} \times \underline{A}$$

$\underline{B} \cdot \underline{n} = 0$ on tube

$$- 2c\eta \int d^3x (\underline{J} \cdot \underline{B})$$

$$= - \int c\eta\, d\underline{S} \cdot \left[\underline{\nabla} \underline{B} \cdot \underline{A} - \cancel{\underline{A} \cdot \underline{\nabla} \underline{B}}^{\,0} \right] - 2c\eta \int d^3x\, \underline{J} \cdot \underline{B}$$

$$= - 2c\eta \int d^3x (\underline{J} \cdot \underline{B})$$

$\Rightarrow$ have shown:

$$\boxed{\frac{dK}{dt} = - 2c\eta \int d^3x (\underline{J} \cdot \underline{B})}$$

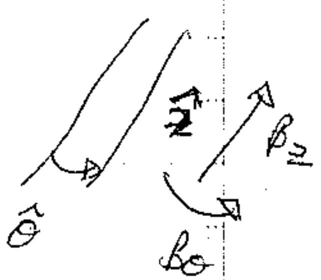
and clearly! $\frac{dK}{dt} \to 0$ as $\eta \to 0$ (non-singular $\underline{J}$)

∴ Magnetic Helicity is conserved in ideal MHD.

→ Magnetic Helicity conserved, but what does it mean?

- helicity is non-trivial ⇒ more than just helical field lines.

interesting to note: $q(r) = \frac{rB_z}{RB_\theta(r)} = \frac{1}{R\mu(r)}$



$\mu(r) = \frac{B_\theta(r)}{rB_z}$ → field line pitch.

cylindrical plasma → $\underline{B} = \underline{B}(r)$

Now, $A_\theta = \frac{1}{r}\int_0^r r\, B_z\, dr$

$A_z = -\int_0^r B_\theta\, dr$

so $\underline{A} \cdot \underline{B} = \frac{B_\theta}{r} \int_0^r r B_z \, dr - B_z \int_0^r B_\theta \, dr$

$= \mu B_z \int_0^r \frac{B_\theta}{\mu} \, dr - B_z \int_0^r B_\theta \, dr$

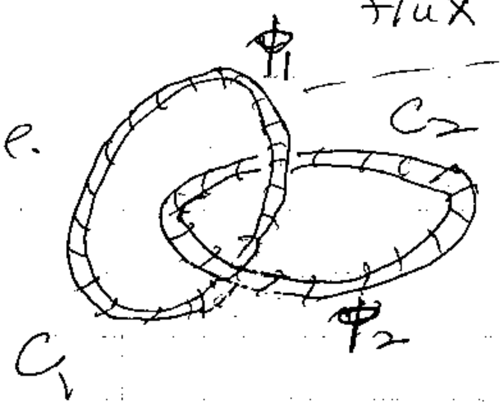
$\underline{A} \cdot \underline{B} = B_z \left[\mu \int_0^r \frac{B_\theta}{\mu} \, dr - \int_0^r B_\theta \, dr \right]$

$= 0$ for constant μ

$\therefore$ non-zero helicity requires $\mu = \mu(r)$
i.e. — pitch varies with radius
$\Rightarrow$ magnetic shear

— physically $\rightarrow$ helicity means self-linkage of flux tubes

i.e.



tube 1: flux
$\Phi = \int d\underline{A} \cdot \underline{B} = \phi_1$
x-section area, const

tube 2: $\Phi = \phi_2$

field in loops, only

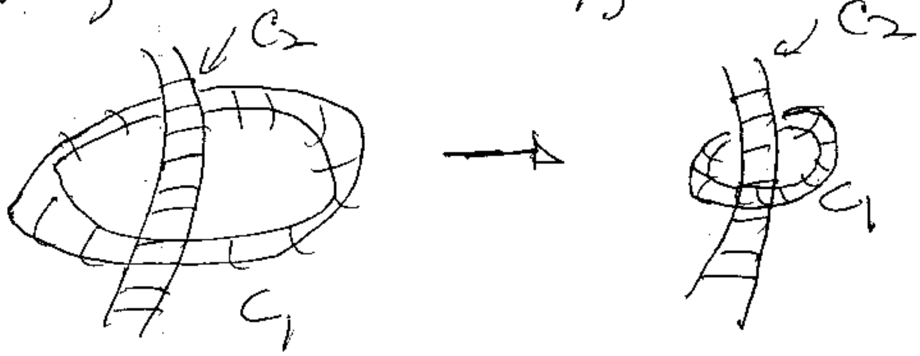
Now, for volume V_1 of tube 1

$$K = \int_{V_1} \underline{A}\cdot\underline{B}\, d^3x = \underbrace{\oint_{C_1}}_{\text{along loop}} d\ell \underbrace{\int_{A_1}}_{\text{x-sect area}} dS\ A\cdot\underline{B}$$

$$= \oint_{C_1} \underline{A}\cdot d\underline{\ell} \int_{S_1} \underline{B}\cdot\hat{n}\ dA$$

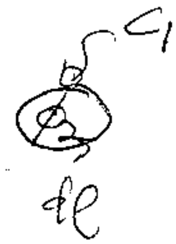
$$= \Phi_1 \oint_{C_1} \underline{A}\cdot d\underline{\ell}$$

Now, can shrink C_1, as no field outside loops



re-oriented

$\Rightarrow$ in x section:



but $\int_{C_1} A\cdot d\ell = \int_{A \text{ enclosed}} B\cdot dS = \Phi_2$

so... $K_1 = \Phi_1 \Phi_2$ → product of fluxes

similarly $K_2 = \Phi_2 \Phi_1$

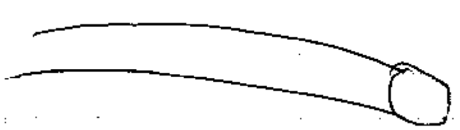
∴ $K = 2\Phi_1 \Phi_2$

if n windings $K = K_1 + K_2 = \pm 2n\Phi_1\Phi_2$.

⇒ helicity is measure of self-linkage of magnetic configuration.

Why care → Taylor Conjecture (1974) (J.B. Taylor)

- in magnetic confinement, of great interest to determine how fields, currents self-organize

- RFP  ~ toroid ~ toroidal current

well fit by
$$B_z = B_0 J_0(\alpha r)$$
$$B_\theta = B_0 J_1(\alpha r)$$

$\underline{J} \times \underline{B} = 0$ ⇒ force free

⇒ why so robust?, especially since RFP so turbulent

- Taylor conjectured conservation of magnetic helicity constrains relaxation to force-free state.

Key Point
- helicity conserved in flux tubes, to η
- toroidal plasma $\Leftrightarrow$ many small tubes

etc.

- recall Sweet-Parker model: magnetic reconnection / resistive dissipation effective on small scales.

$\Rightarrow$ Taylor Conjecture: At finite η, helicity of small tubes dissipated but, global helicity conserved.

i.e.

$$\underbrace{\int}_{\text{plasma volume}} \underline{A} \cdot \underline{B} \, d^3x = K_0 \rightarrow \text{conserved.}$$

$\therefore$ Taylor conjectured that actual magnetic configuration could be explained by minimum principle:

$$\delta\left[\int_V d^3x\,\frac{B^2}{8\pi} + \lambda\int_V d^3x\,\underline{A}\cdot\underline{B}\right] = 0$$

i.e. minimize magnetic energy subject to constraint of conserved global helicity,

Comments:

→ it works! - indeed amazingly well - for RFPs, spheromaks, etc. Departures only recently being discovered

→ inspired idea of helicity injection as way to maintain configurations.

→ it is a conjecture → no proof.

Hypothesis: Selective Decay → energy cascades → small scale; helicity cascades → large scale (less dissipation)

- relevance to driven system?
i.e. in real RFP, transformer on.

→ dynamics? – how does relaxation occur

→ more in discussion of kinks, tearing.